

# INTEGRAL MEAN ESTIMATES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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**ABSTRACT.** Let  $P(z)$  be a polynomial of degree  $n$  having all zeros in  $|z| \leq k$  where  $k \leq 1$ , then it was proved by Dewan *et al* [6] that for every real or complex number  $\alpha$  with  $|\alpha| \geq k$  and each  $r \geq 0$

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \text{Max}_{|z|=1} |D_\alpha P(z)|.$$

In this paper, we shall present a refinement and generalization of above result and also extend it to the class of polynomials  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , having all its zeros in  $|z| \leq k$  where  $k \leq 1$  and thereby obtain certain generalizations of above and many other known results.

## 1. Introduction and statement of results

Let  $P(z)$  be a polynomial of degree  $n$ . It was shown by Turán [12] that if  $P(z)$  has all its zeros in  $|z| \leq 1$ , then

$$\text{Max}_{|z|=1} |P(z)| \leq 2 \text{Max}_{|z|=1} |P'(z)|. \quad (1.1)$$

Inequality (1.1) is best possible with equality holds for  $P(z) = \alpha z^n + \beta$  where  $|\alpha| = |\beta|$ . The above inequality (1.1) of Turán [12] was generalized by Malik [10], who proved that if  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then

$$\text{Max}_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \text{Max}_{|z|=1} |P(z)|. \quad (1.2)$$

where as for  $k \geq 1$ , Govil [7] showed that

$$\text{Max}_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \text{Max}_{|z|=1} |P(z)|, \quad (1.3)$$

Both the above inequalities (1.2) and (1.3) are best possible, with equality in (1.2) holding for  $P(z) = (z+k)^n$ , where  $k \geq 1$ . While in (1.3) the equality holds for the polynomial  $P(z) = \alpha z^n + \beta k^n$  where  $|\alpha| = |\beta|$ .

As a refinement of (1.2), Aziz and Shah [4] proved if  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then

$$\text{Max}_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \left\{ \text{Max}_{|z|=1} |P(z)| + \frac{1}{k^{n-1}} \text{Min}_{|z|=1} |P(z)| \right\}. \quad (1.4)$$

Let  $D_\alpha P(z)$  denotes the polar derivative of the polynomial  $P(z)$  of degree  $n$  with respect to the point  $\alpha$ , then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial  $D_\alpha P(z)$  is a polynomial of degree at most  $n-1$  and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \left[ \frac{D_\alpha P(z)}{\alpha} \right] = P'(z).$$

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Aziz and Rather [2] extends (1.2) to polar derivatives of a polynomial and proved that if all the zeros of  $P(z)$  lie in  $|z| \leq k$  where  $k \leq 1$  then for every real or complex number  $\alpha$  with  $|\alpha| \geq k$ ,

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - k}{1 + k} \right) \max_{|z|=1} |P(z)|. \quad (1.5)$$

For the class of polynomials  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , of degree  $n$  having all its zeros in  $|z| \leq k$  where  $k \leq 1$ , Aziz and Rather [3] proved that if  $\alpha$  is real or complex number with  $|\alpha| \geq k^\mu$  then

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - k^\mu}{1 + k^\mu} \right) \max_{|z|=1} |P(z)|. \quad (1.6)$$

Malik [11] obtained a generalization of (1.1) in the sense that the left-hand side of (1.1) is replaced by a factor involving the integral mean of  $|P(z)|$  on  $|z| = 1$ . In fact he proved that if  $P(z)$  has all its zeros in  $|z| \leq 1$ , then for each  $q > 0$ ,

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \quad (1.7)$$

If we let  $q$  tend to infinity in (1.7), we get (1.1).

The corresponding generalization of (1.2) which is an extension of (1.7) was obtained by Aziz [1] by proving that if  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$  where  $k \geq 1$ , then for each  $q \geq 1$

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \quad (1.8)$$

The result is best possible and equality in (1.5) holds for the polynomial  $P(z) = \alpha z^n + \beta k^n$  where  $|\alpha| = |\beta|$ .

As a generalization of inequality 1.5, Dewan *et al* [6] obtained an  $L^p$  inequality for the polar derivative of a polynomial and proved the following:

**Theorem 1.1.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex number  $\alpha$  with  $|\alpha| \geq k$  and for each  $r > 0$ ,*

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |D_\alpha P(z)|. \quad (1.9)$$

In this paper, we consider the class of polynomials  $P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}$ ,  $1 \leq \mu \leq n$ , having all its zeros in  $|z| \leq k$  where  $k \leq 1$  and establish some improvements and generalizations of inequalities (1.1), (1.2), (1.5), (1.8) and (1.9).

In this direction, we first present the following interesting results which yields (1.9) as a special case.

**Theorem 1.2.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex  $\alpha, \beta$  with  $|\alpha| \geq k$ ,  $|\beta| \leq 1$  and for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m}{k^{n-1}} \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}} \quad (1.10)$$

where  $m = \min_{|z|=k} |P(z)|$ .

If we take  $\beta = 0$ , we get the following result.

**Corollary 1.3.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex  $\alpha$ , with  $|\alpha| \geq k$  and for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}. \quad (1.11)$$

**Remark 1.4.** Theorem 1.1 follows from (1.11) by letting  $q \rightarrow \infty$  (so that  $p \rightarrow 1$ ) in Corollary 1.3. If we divide both sides of inequality (1.11) by  $|\alpha|$  and make  $\alpha \rightarrow \infty$ , we get (1.5).

Dividing the two sides of (1.10) by  $|\alpha|$  and letting  $|\alpha| \rightarrow \infty$ , we get the following result.

**Corollary 1.5.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex  $\beta$  with  $|\beta| \leq 1$  and for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m}{k^{n-1}} \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}} \quad (1.12)$$

where  $m = \min_{|z|=k} |P(z)|$ .

If we let  $q \rightarrow \infty$  in (1.12), we get the following corollary.

**Corollary 1.6.** *If  $P(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex  $\beta$  with  $|\beta| \leq 1$  and for each  $r > 0$ , we have*

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m}{k^{n-1}} \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |P'(z)|, \quad (1.13)$$

where  $m = \min_{|z|=k} |P(z)|$ .

**Remark 1.7.** If we let  $r \rightarrow \infty$  in (1.13) and choosing argument of  $\beta$  suitably with  $|\beta| = 1$ , we obtain (1.4).

Next, we extend (1.9) to the class of polynomials  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , having all its zeros in  $|z| \leq k$ ,  $k \leq 1$  and thereby obtain the following result.

**Theorem 1.8.** *If  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$  where  $k \leq 1$ , then for every real or complex  $\alpha$  with  $|\alpha| \geq k^\mu$  and for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}. \quad (1.14)$$

**Remark 1.9.** We let  $r \rightarrow \infty$  and  $p \rightarrow \infty$  (so that  $q \rightarrow 1$ ) in (1.14), we get inequality (1.6).

If we divide both sides of (1.14) by  $|\alpha|$  and make  $\alpha \rightarrow \infty$ , we get the following result.

**Corollary 1.10.** *If  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$  where  $k \leq 1$ , then for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}. \quad (1.15)$$

Letting  $q \rightarrow \infty$  (so that  $p \rightarrow 1$ ) in (1.14), we get the following result:

**Corollary 1.11.** *If  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , where  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex number  $\alpha$  with  $|\alpha| \geq k^\mu$  and for each  $r > 0$ ,*

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |D_\alpha P(z)|. \quad (1.16)$$

As a generalization of Theorem 1.8, we present the following result:

**Theorem 1.12.** *If  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$  where  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$  where  $k \leq 1$ , then for every real or complex  $\alpha$  with  $|\alpha| \geq k^\mu$  and for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}} \quad (1.17)$$

where  $m = \min_{|z|=k} |P(z)|$ .

If we divide both sides by  $|\alpha|$  and make  $\alpha \rightarrow \infty$ , we get the following result:

**Corollary 1.13.** *If  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$  where  $k \leq 1$ , then for each  $r > 0$ ,  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ , we have*

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}} \quad (1.18)$$

where  $m = \min_{|z|=k} |P(z)|$ .

Letting  $q \rightarrow \infty$  (so that  $p \rightarrow 1$ ) in (1.14), we get the following result:

**Corollary 1.14.** *If  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$  where  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ , where  $k \leq 1$ , then for every real or complex number  $\alpha$  with  $|\alpha| \geq k^\mu$  and for each  $r > 0$ ,*

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |D_\alpha P(z)| \quad (1.19)$$

where  $m = \min_{|z|=k} |P(z)|$ .

## 2. Lemmas

For the proofs of the theorems, we need the following Lemmas:

**Lemma 2.1.** *If  $P(z)$  is a polynomial of degree almost  $n$  having all its zeros in  $|z| \leq k$   $k \leq 1$  then for  $|z| = 1$ ,*

$$|Q'(z)| + \frac{nm}{k^{n-1}} \leq k|P'(z)|, \quad (2.1)$$

where  $Q(z) = z^n \overline{P(1/\bar{z})}$  and  $m = \min_{|z|=k} |P(z)|$ .

The above Lemma is due to Govil and McTume [8].

**Lemma 2.2.** *Let  $P(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$ ,  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$ , which does not vanish for  $|z| < k$ , where  $k \geq 1$  then for  $|z| = 1$ ,*

$$k^\mu |P'(z)| \leq |Q'(z)|, \quad (2.2)$$

where  $Q(z) = z^n \overline{P(1/\bar{z})}$ .

The above Lemma is due to Chan and Malik [5]. By applying Lemma 2.2 to the polynomial  $z^n \overline{P(1/\bar{z})}$ , one can easily deduce:

**Lemma 2.3.** *Let  $P(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}$ ,  $1 \leq \mu \leq n$ , is a polynomial of degree  $n$ , having all its zeros in  $|z| \leq k$ , where  $k \leq 1$  then for  $|z| = 1$*

$$k^\mu |P'(z)| \geq |Q'(z)|, \quad (2.3)$$

where  $Q(z) = z^n \overline{P(1/\bar{z})}$ .

### 3. Proof of Theorems

**Proof of Theorem 1.2.** Let  $Q(z) = z^n \overline{P(1/\bar{z})}$  then  $P(z) = z^n \overline{Q(1/\bar{z})}$  and it can be easily verified that for  $|z| = 1$ ,

$$|Q'(z)| = |nP(z) - zP'(z)| \quad \text{and} \quad |P'(z)| = |nQ(z) - zQ'(z)|. \quad (3.1)$$

By Lemma (2.1), we have for every  $\beta$  with  $|\beta| \leq 1$  and  $|z| = 1$ ,

$$\left| Q'(z) + \beta \frac{nmz^{n-1}}{k^{n-1}} \right| \leq |Q'(z)| + \frac{nm}{k^{n-1}} \leq k|P'(z)|. \quad (3.2)$$

Using (3.1) in (3.2), for  $|z| = 1$  we have

$$\left| Q'(z) + \beta \frac{nmz^{n-1}}{k^{n-1}} \right| \leq k|nP(z) - zP'(z)|. \quad (3.3)$$

By Lemma 2.3 with  $\mu = 1$ , for every real or complex number  $\alpha$  with  $|\alpha| \geq k$  and  $|z| = 1$ , we have

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha| |P'(z)| - |Q'(z)| \\ &\geq (|\alpha| - k) |P'(z)|. \end{aligned} \quad (3.4)$$

Since  $P(z)$  has all its zeros in  $|z| \leq k \leq 1$ , it follows by Gauss-Lucas Theorem that all the zeros of  $P'(z)$  also lie in  $|z| \leq k \leq 1$ . This implies that the polynomial

$$z^{n-1} \overline{P'(1/\bar{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in  $|z| < 1$ . Therefore, it follows from (3.3) that the function

$$w(z) = \frac{z \left( Q'(z) + \beta \frac{nmz^{n-1}}{k^{n-1}} \right)}{k(nQ(z) - zQ'(z))}$$

is analytic for  $|z| \leq 1$  and  $|w(z)| \leq 1$  for  $|z| = 1$ . Furthermore,  $w(0) = 0$ . Thus the function  $1 + kw(z)$  is subordinate to the function  $1 + kz$  for  $|z| \leq 1$ . Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi} |1 + kw(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta, \quad r > 0. \quad (3.5)$$

Now

$$1 + kw(z) = \frac{n \left( Q(z) + \beta \frac{mz^n}{k^{n-1}} \right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1} \overline{P'(1/\bar{z})}| = |nP(z) - zQ'(z)|, \quad \text{for } |z| = 1,$$

therefore for  $|z| = 1$ ,

$$n \left| Q(z) + \bar{\beta} \frac{mz^n}{k^{n-1}} \right| = |1 + kw(z)| |nQ(z) - zQ'(z)| = |1 + kw(z)| |P'(z)|.$$

equivalently,

$$n \left| z^n \overline{P(1/\bar{z})} + \bar{\beta} \frac{mz^n}{k^{n-1}} \right| = |1 + kw(z)| |P'(z)|.$$

This implies

$$n \left| P(z) + \beta \frac{m}{k^{n-1}} \right| = |1 + kw(z)| |P'(z)| \quad \text{for } |z| = 1. \quad (3.6)$$

From (3.4) and (3.6), we deduce that for  $r > 0$ ,

$$n^r (|\alpha| - k)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m}{k^{n-1}} \right|^r d\theta \leq \int_0^{2\pi} |1 + kw(e^{i\theta})|^r |D_\alpha P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality and using (3.5), for  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ ,

$$n^r (|\alpha| - k)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m}{k^{n-1}} \right|^r d\theta \leq \left( \int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta \right)^{1/p} \left( \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right)^{1/q},$$

equivalently,

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m}{k^{n-1}} \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}$$

which proves the desired result.  $\square$

**Proof of Theorem 1.8.** Since  $P(z)$  has all its zeros in  $|z| \leq k$ , therefore, by using Lemma 2.3 we have for  $|z| = 1$ ,

$$|Q'(z)| \leq k^\mu |nQ(z) - zQ'(z)|. \quad (3.7)$$

Now for every real or complex number  $\alpha$  with  $|\alpha| \geq k^\mu$ , we have

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geq |\alpha| |P'(z)| - |nP(z) - zP'(z)|, \end{aligned}$$

by using (3.1) and Lemma 2.3, for  $|z| = 1$ , we get

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha| |P'(z)| - |Q'(z)| \\ &\geq (|\alpha| - k^\mu) |P'(z)|. \end{aligned} \quad (3.8)$$

Since  $P(z)$  has all its zeros in  $|z| \leq k \leq 1$ , it follows by Gauss-Lucas Theorem that all the zeros of  $P'(z)$  also lie in  $|z| \leq k \leq 1$ . This implies that the polynomial

$$z^{n-1} \overline{P'(1/\bar{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in  $|z| < 1$ . Therefore, it follows from (3.7) that the function

$$w(z) = \frac{zQ'(z)}{k^\mu (nQ(z) - zQ'(z))}$$

is analytic for  $|z| \leq 1$  and  $|w(z)| \leq 1$  for  $|z| = 1$ . Furthermore,  $w(0) = 0$ . Thus the function  $1 + k^\mu w(z)$  is subordinate to the function  $1 + k^\mu z$  for  $|z| \leq 1$ . Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta, \quad r > 0. \quad (3.9)$$

Now

$$1 + k^\mu w(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1} \overline{P'(1/\bar{z})}| = |nQ(z) - zQ'(z)|, \text{ for } |z| = 1,$$

therefore, for  $|z| = 1$ ,

$$n|Q(z)| = |1 + k^\mu w(z)| |nQ(z) - zQ'(z)| = |1 + k^\mu w(z)| |P'(z)|. \quad (3.10)$$

From (3.8) and (3.10), we deduce that for  $r > 0$ ,

$$n^r (|\alpha| - k^\mu)^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^r |D_\alpha P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality and (3.9), for  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ ,

$$n^r (|\alpha| - k^\mu)^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \left( \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^{pr} d\theta \right)^{1/p} \left( \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right)^{1/q},$$

equivalently,

$$n (|\alpha| - k^\mu) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}$$

which proves the desired result.  $\square$

**Proof of Theorem 1.12.** Let  $m = \min_{|z|=k} |P(z)|$ , so that  $m \leq |P(z)|$  for  $|z| = k$ . If  $P(z)$  has a zero on  $|z| = k$  then  $m = 0$  and result follows from Theorem 1.8. Henceforth we suppose that all the zeros of  $P(z)$  lie in  $|z| < k$ . Therefore for every  $\beta$  with  $|\beta| < 1$ , we have  $|m\beta| < |P(z)|$  for  $|z| = k$ . Since  $P(z)$  has all its zeros in  $|z| < k \leq 1$ , it follows by Rouché's theorem that all the zeros of  $F(z) = P(z) + \beta m$  lie in  $|z| < k \leq 1$ . If  $G(z) = z^n \overline{P'(1/\bar{z})} = Q(z) + \bar{\beta} m z^n$ , then by applying Lemma 2.3 to polynomial  $F(z) = P(z) + \beta m$ , we have for  $|z| = 1$ ,

$$|G'(z)| \leq k^\mu |F'(z)|.$$

This gives

$$|Q'(z) + nm\bar{\beta}z^{n-1}| \leq k^\mu |P'(z)|. \quad (3.11)$$

Using (3.1) in (3.11), for  $|z| = 1$  we have

$$|Q'(z) + nm\bar{\beta}z^{n-1}| \leq k^\mu |nQ(z) - zQ'(z)| \quad (3.12)$$

Since  $P(z)$  has all its zeros in  $|z| < k \leq 1$ , it follows by Gauss-Lucas Theorem that all the zeros of  $P'(z)$  also lie in  $|z| < k \leq 1$ . This implies that the polynomial

$$z^{n-1} \overline{P'(1/\bar{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in  $|z| < 1$ . Therefore, it follows from (3.12) that the function

$$w(z) = \frac{z(Q'(z) + nm\bar{\beta}z^{n-1})}{k^\mu (nQ(z) - zQ'(z))}$$

is analytic for  $|z| \leq 1$  and  $|w(z)| \leq 1$  for  $|z| = 1$ . Furthermore,  $w(0) = 0$ . Thus the function  $1 + k^\mu w(z)$  is subordinate to the function  $1 + k^\mu z$  for  $|z| \leq 1$ . Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta, \quad r > 0. \quad (3.13)$$

Now

$$1 + k^\mu w(z) = \frac{n(Q(z) + m\bar{\beta}z^n)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1}\overline{P'(1/\bar{z})}| = |nQ(z) - zQ'(z)|, \text{ for } |z| = 1,$$

therefore, for  $|z| = 1$ ,

$$n|Q(z) + m\bar{\beta}z^n| = |1 + k^\mu w(z)||nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|.$$

This implies

$$n|G(z)| = |1 + k^\mu w(z)||nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|. \quad (3.14)$$

Since  $|F(z)| = |G(z)|$  for  $|z| = 1$ , therefore, from (3.14) we get

$$n|P(z) + \beta m| = |1 + k^\mu w(z)||P'(z)| \text{ for } |z| = 1. \quad (3.15)$$

From (3.8) and (3.15), we deduce that for  $r > 0$ ,

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \leq \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^r |D_\alpha P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality in conjunction with (3.13) for  $p > 1$ ,  $q > 1$  with  $p^{-1} + q^{-1} = 1$ ,

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \leq \left( \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^{pr} d\theta \right)^{1/p} \left( \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right)^{1/q},$$

equivalently,

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{qr}}$$

which proves the desired result.  $\square$

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